Local Central Limit Theorem for Long-Range Two-Body Potentials at Sufficiently High Temperatures

In just 3 slides

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- **Goal:** *Local Central Limit Theorem* for long-range interactions.
- **Model:** Translation-invariant pair potential $\Phi = \{\Phi_{\{x,y\}}\}_{x \neq y}$ that is *absolutely summable* $\sup_{x \in \mathbb{Z}^d} \sum_{\substack{y \in \mathbb{Z}^d \\ y \neq x}} \|\Phi_{\{x,y\}}\| < \infty$
- Define the Hamiltonian in a finite set Λ with boundary condition $\omega \in \Omega$ associated to the potential Φ by

$$H_\Lambda^\omega(\sigma) = \sum_{\substack{x,y \in \Lambda \\ x \neq y}} \Phi_{\{x,y\}}(\sigma) + \sum_{x \in \Lambda} \sum_{y \notin \Lambda} \Phi_{\{x,y\}}(\sigma_\Lambda \omega_{\Lambda^c}), \qquad (1)$$

where the configuration $\left((\sigma_\Lambda \omega_{\Lambda^c})_x\right)_{x \in \mathbb{Z}^d}$ means

$$(\sigma_\Lambda \omega_{\Lambda^c})_x = \begin{cases} \sigma_x, & \text{if } x \in \Lambda, \\ \omega_x, & \text{if } x \notin \Lambda. \end{cases} \qquad (2)$$

- For inverse temperature $\beta > 0$, we define

$$\mu_{\Lambda,\beta}^\omega(\sigma) = \frac{e^{-\beta H_\Lambda^\omega(\sigma)}}{Z_{\Lambda,\beta}^\omega}, \qquad (3)$$

where $Z_{\Lambda,\beta}^\omega$ is the partition function given by

$$Z_{\Lambda,\beta}^\omega = \int_{E^\Lambda} e^{-\beta H_\Lambda^\omega(\sigma)} \prod_{x \in \Lambda} \lambda(d\sigma_x). \qquad (4)$$

- **Observable (lattice-valued):** Let $f : \Omega \to \mathbb{Z}$ be bounded.

$$S_k := \sum_{x \in \Lambda_k} f \circ \theta_x, \qquad D_k := \mathrm{Var}_{\mu^{\omega_k}_{\Lambda_k,\beta}}(S_k), \qquad \bar{S}_k := \frac{S_k - \mu^{\omega_k}_{\Lambda_k,\beta}(S_k)}{\sqrt{D_k}}.$$

- By $f \circ \theta_x$ is lattice distributed we mean that we pick the same $a \in \mathbb{Z}$ and h for every $x \in \mathbb{Z}^d$.
- If f is bounded, then the possible values for f are $a + bh$ where b is in the set $\{p, p+1, \dots, q\}$ for some integers $p < q$. Then S_k has possible values $|\Lambda_k|a + bh$, where $b \in B_k := \{p|\Lambda_k|, \dots, q|\Lambda_k|\}$.
- **Local CLT**

$$z_{k,b} := \frac{|\Lambda_k|a + bh - \mu^{\omega_k}_{\Lambda_k,\beta}(S_k)}{\sqrt{D_k}},$$

$$\sup_{b \in B_k} \left| \frac{\sqrt{D_k}}{h} \mu^{\omega_k}_{\Lambda_k,\beta}(S_k = |\Lambda_k|a + bh) - \frac{1}{\sqrt{2\pi}} e^{-z_{k,b}^2/2} \right| \xrightarrow[k \to \infty]{} 0.$$

Main result (informal): If the integral CLT holds for $(\mu^{\omega_k}_{\Lambda_k,\beta})_{k \geq 1}$, then the local CLT holds as well, provided β is sufficiently small (high temperature).

We estimate

$$
\begin{aligned}
&\sup_{b\in B_k} 2\pi \left| \frac{\sqrt{D_k}}{h} \mu^{\omega_k}_{\Lambda_k,\beta}(S_k = |\Lambda_k| a + bh) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_{k,b}^2}{2}\right) \right| \\
&\leq \int_{-B}^{B} \left| \mu^{\omega_k}_{\Lambda_k,\beta}(\exp(it\overline{S}_k)) - \exp(-t^2/2) \right| \, dt + \int_{|t|\geq B} \exp(-t^2/2)\, dt \\
&\quad + \int_{B<|t|<\delta\sqrt{D_k}} \left| \mu^{\omega_k}_{\Lambda_k,\beta}(\exp(it\overline{S}_k)) \right| \, dt \\
&\quad + \int_{\delta\sqrt{D_k}\leq|t|\leq\frac{\pi}{h}\sqrt{D_k}} \left| \mu^{\omega_k}_{\Lambda_k,\beta}(\exp(it\overline{S}_k)) \right| \, dt. \qquad (5)
\end{aligned}
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- *Third term:* Taylor Reminder Theorem (along with Theorem 3).
- *Fourth term:* Cluster expansion at high temperatures (Section 3).
- **What can we learn?**A way to prove Local CLT for the **challenging long-range models.**